Phases of a Stack of Membranes at Large-d

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The phase diagram of a stack of tensionless membranes with nonlinear curvature energy and vertical harmonic interaction is calculated exactly in a large number of dimensions of configuration space. At low temperatures, the system forms a lamellar phase with spontaneously broken translational symmetry in the vertical direction. At a critical temperature, the stack disorders vertically in a melting-like transition. The critical temperature is determined as a function of the interlayer separation l.

I. INTRODUCTION

Recently, a model for a finite stack of tensionless membranes [1] was studied with respect to the effects of higher-order terms of the curvature energy [2]. The approach was perturbative, using the renormalization group to sum infinitely many terms. It was shown that thermal fluctuations induce the melting of the stack into a vertically disordered phase. By its nature, the perturbative expansion was able to give a satisfactory description only for the ordered phase.

For a description of the disordered phase and a better understanding of the entire transition, we analyze in this paper the behavior of a stack of tensionless membranes exactly for very large dimension d of the embedding space. Since the model is exactly solvable in this limit, we can calculate all its relevant properties explicitly, in particular its complete phase diagram as a function of the interlayer separation l.

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II. THE MODEL

As in Ref. [2], we consider a model in which a multilayer system is made up of $(N+1)$ fluid membranes, parallel to the xy-plane of a Cartesian coordinate system, separated a distance l. If the vertical displacement of the mth membrane with respect to this reference plane is described by a function $u_m(\mathbf{x}) \equiv u(\mathbf{x}_\perp, ml)$, where $\mathbf{x}_\perp = (x, y)$, the energy of the stack reads:

$$E = \sum_m \int \mathrm{d}^2 x_\perp \sqrt{g_m} \left[\tfrac{1}{2}\kappa_0 H_m^2 + \frac{B_0}{2l}(u_m - u_{m-1})^2 \right]. \tag{1}$$

Here, $H_m = \partial_a N_{m,i}$ is the mean curvature, where $\mathbf{N}_m \propto (-\partial_1 u_m, -\partial_2 u_m, 1)$ is the unit normal to the mth membrane, and

$$g_{m,ij} = \delta_{ij} + \partial_i u_m \partial_j u_m \tag{2}$$

the induced metric, with $i, j = 1, 2$, $\partial_1 = \partial/\partial x$, $\partial_2 = \partial/\partial y$ and $g_m = \det[g_{m,ij}]$. The parameter κ_0 is the bending rigidity of a single membrane, and B_0 the compressibility of the stack. In Eq. (1), as in the following, the subscript 0 denotes bare quantities, whereas renormalized parameters will carry no subscript.

For slow spatial variations, the discrete variable ml may be replaced with a continuous one, and $u(\mathbf{x}_\perp, ml) \to u(\mathbf{x})$, where $\mathbf{x} = (\mathbf{x}_\perp, z)$. In this limit, the energy (1) reduces to

$$E = \int_0^{L_\parallel} \mathrm{d}z \int \mathrm{d}^2 x_\perp \sqrt{g} \left[\tfrac{1}{2}K_0 H^2 + \tfrac{1}{2}B_0(\partial_z u)^2 \right]. \tag{3}$$

Here we have introduced the bulk version of the bending rigidity $K_0 \equiv \kappa_0/l$, and defined $L_\parallel \equiv Nl$. The gradient energy $(\partial_z u)^2$ has the unphysical feature that it gives z-dependent reparametrizations a kinetic energy. It should therefore be replaced by the normal gradient energy $(\mathbf{N} \cdot \nabla u)^2$. We have seen in Ref. [2] that this changes the critical exponent with which the renormalized B vanishes as the bending rigidity becomes critical. However, in the limit of large d, which we are going to investigate, the difference between the two gradient energies will be negligible.

III. LARGE-d ANALYSIS

For arbitrary d, the vertical displacement of the mth membrane in the stack becomes a $(d-2)$-vector field $\mathbf{u}_m(\mathbf{x}_\perp)$.

A. Partition function and the energy

It is useful to consider g_{ij} as an independent field [3], and impose relation (2) with help of a Lagrange multiplier λ_{ij}. We write the partition function as a functional integral over all possible configurations $\mathbf{u}_m(\mathbf{x}_\perp)$ of the individual membranes in the stack, as well as over all possible metrics $g_{m,ij}$. After taking again the continuum limit, the partition function reads:

$$Z = \int \mathcal{D}g \, \mathcal{D}\lambda \, \mathcal{D}\mathbf{u} \, \mathrm{e}^{-E_0/k_\mathrm{B}T}, \tag{4}$$

with

$$E_0 = \int \mathrm{d}z \, \mathrm{d}^2 x_\perp \sqrt{g} \left\{ \sigma_0 + \tfrac{1}{2}B_0(\partial_z \mathbf{u})^2 + \tfrac{1}{2}K_0(\partial_k^2 \mathbf{u})^2 + \tfrac{1}{2}K_0 \lambda^{ij}(\delta_{ij} + \partial_i \mathbf{u}\partial_j \mathbf{u} - g_{ij}) - \tfrac{1}{4}\tau_0 \lambda_{ii}^2 \right\}, \tag{5}$$

where $\mathbf{u}$ is a $(d-2)$-dimensional vector-function of $\mathbf{x}_\perp, z$. Note that the functional integral over λ in (4) has to be performed along the imaginary axis to result in a δ-function. We have also introduced a term proportional to λ_{ii}^2. This term is necessary to renormalize the theory, and its coefficient τ_0 corresponds to the large-d in-plane compressibility of the membranes. Since we take the membranes to be incompressible, we shall set the renormalized τ equal to zero at the end of our calculations. We have also included a surface tension σ_0, again to absorb infinities and to be set equal to zero after renormalization.

The functional integral over $\mathbf{u}$ in Eq. (4) is Gaussian and can be carried out to yield an effective energy

$$E_\mathrm{eff} = \tilde{E}_0 + E_1, \tag{6}$$

with

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$$\tilde{E}_0 = \int \mathrm{d}z \, \mathrm{d}^2 x_\perp \sqrt{g} \left[\sigma_0 + \tfrac{1}{2} K_0 \lambda^{ij} (\delta_{ij} - g_{ij}) - \tfrac{1}{4} \tau_0 \lambda_{ii}^2 \right], \tag{7}$$

and

$$E_1 = \frac{d-2}{2} k_{\mathrm{B}} T \, \mathrm{Tr} \ln \left[B_0 \omega^2 + K_0 (q_\perp^4 - q_i \lambda^{ij} q_j) \right], \tag{8}$$

where the functional trace Tr is here an integral over space as well as the integral over wavevectors $\mathbf{q}_\perp$ and ω, after replacing $\partial_z^2 \to -\omega^2$ and $g^{ij} \partial_i \partial_i \to -\mathbf{q}^2$. Note that the discrete nature of the stack restricts the integral over the wavevectors ω to the first Brillouin zone $|\omega| < \pi/l$.

In the large-d limit, the partition function (4) is dominated by the saddle point of the effective energy (6) with respect to the metric g_{ij} and the Lagrange multiplier λ^{ij}. For very large membranes, the saddle point can be assumed to be symmetric and homogeneous [4–6]:

$$g_{ij} = \varrho_0 \delta_{ij}; \ \lambda^{ij} = \lambda_0 g^{ij} = \frac{\lambda_0}{\varrho_0} \delta^{ij}, \tag{9}$$

with constant ϱ_0 and λ_0. At the saddle point the effective energy (6) becomes the free energy of the system.

In the following we shall investigate both the case of an infinite and a finite stack of membranes. As we will see, the large-d approximation allows for the vertical melting even in an infinite stack, which is not found perturbatively [1].

B. Infinite stack

Let us first analyze the case of an infinite stack. To simplify our calculations, we assume the number $N + 1$ of membranes in the stack to be very large, making the distance l between them very small. In this regime, we may extend the limits $\pm \pi/l$ of the integral over ω to infinity. The explicit l-dependence will be introduced later into our calculations.

After evaluating the functional trace in Eq. (8), we obtain

$$E_1 = \frac{d k_{\mathrm{B}} T}{2} \int \mathrm{d}z \mathrm{d}^2 x_\perp \varrho_0 \sqrt{\frac{K_0}{B_0}} \left\{ \frac{\Lambda^4}{8\pi} + \frac{\lambda_0}{8\pi} \Lambda^2 + \frac{\lambda_0^2}{64\pi} \left[1 - 2 \ln \left(\frac{4\Lambda^2}{\lambda_0} \right) \right] \right\}, \tag{10}$$

where ultraviolet divergences are regularized by introducing a sharp transverse wavevector cutoff Λ and $d - 2$ has been replaced by d for large d.

We may now absorb the first term in (10) by renormalizing σ_0, so that

$$\sigma = \sigma_0 + \frac{dk_{\mathrm{B}}T}{16\pi}\sqrt{\frac{K_0}{B_0}}\Lambda^4 \tag{11}$$

is the physical surface tension, which is set equal to zero. The second, quadratically divergent term in (10) is used to define the critical temperature as

$$\frac{1}{T_{\mathrm{c}}} \equiv \frac{dk_{\mathrm{B}}}{16\pi}\frac{\Lambda^2}{\sqrt{BK}}. \tag{12}$$

The next divergent term, proportional to λ_0^2, is regularized by introducing a renormalization scale μ and modifying the the in-plane compressibility to

$$\tau = \tau_0 + \frac{dk_{\mathrm{B}}T}{32\pi}\sqrt{\frac{K_0}{B_0}}\ln\left(4e^{-1/2}\frac{\Lambda^2}{\mu^2}\right). \tag{13}$$

The physical in-plane compressibility τ is now set equal to zero, as explained in the previous section.

The effective energy thus becomes

$$E_{\mathrm{eff}} = \int \mathrm{d}z\,\mathrm{d}^2x_\perp K\lambda\varrho\left\{\left(\frac{1}{\varrho}-1\right) + \frac{T}{T_{\mathrm{c}}} + \frac{aT}{\sqrt{K}}\lambda\left[\ln\left(\frac{\lambda}{\bar{\lambda}}\right) - \frac{1}{2}\right]\right\}, \tag{14}$$

with the constants $a \equiv dk_{\mathrm{B}}/64\pi\sqrt{B}$, $\bar{\lambda} \equiv \mu^2 e^{-1/2}$.

From the second derivative matrix of E_{eff} with respect to ϱ and λ we find that the stability of the saddle point is guaranteed only for $\lambda < \bar{\lambda}$.

Extremizing the above expression with respect to ϱ, we find two solutions for λ, namely $\lambda = 0$ and $\lambda = \lambda_\infty$, with

$$\lambda_\infty\left[\ln\left(\frac{\lambda_\infty}{\bar{\lambda}}\right) - \frac{1}{2}\right] = \frac{\sqrt{K}}{a}\left(\frac{1}{T} - \frac{1}{T_{\mathrm{c}}}\right). \tag{15}$$

For $T < T_{\mathrm{c}}$, this equation has no solution for λ_∞. In this case, the only possible solution is $\lambda = 0$, which corresponds to the ordered phase as we shall verify later. For $T > T_{\mathrm{c}}$, the saddle point lies at $\lambda = \lambda_\infty$, which is now well-defined. This is the vertically disordered phase.

The free energy density at the extremum is given by

$$f = K\lambda_\infty \tag{16}$$

and its behavior is similar to the one found perturbatively in Ref. [2]. (see Fig. 1).
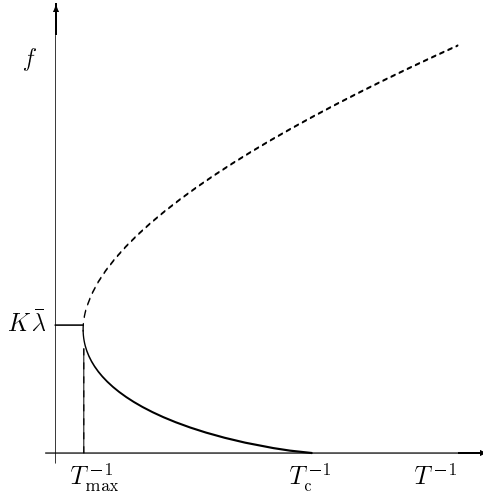
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FIG. 1. Free energy density of an infinite stack of membranes. The dashed curve indicates the unphysical branch of the solution of Eq. (15).

Extremizing the effective energy (14) with respect to λ, we find ϱ as a function of temperature. For $T < T_{\mathrm{c}}$ it is given by

$$\varrho_-^{-1} = 1 - \frac{T}{T_{\mathrm{c}}}. \tag{17}$$

This as T approaches T_{c} from below, indicating the vertical melting at T_{c}. In the disordered phase, ϱ is found to be

$$\varrho_+^{-1} = \frac{T}{T_{\mathrm{c}}} - 1 - \frac{a\lambda_\infty T}{\sqrt{K}}. \tag{18}$$

As T approaches T_{c} from above, λ_∞ tends to zero, and ϱ goes again to infinity.

The positivity of ϱ and the stability of the saddle point imply that there is a maximum temperature, given by

$$\frac{1}{T_{\mathrm{max}}} = \frac{1}{T_{\mathrm{c}}} - \frac{a\bar{\lambda}}{\sqrt{K}}, \tag{19}$$

below which our assumption that the membranes in the stack are in-plane incompressible does not lead to a stable system.

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C. Finite stack of many membranes

Let us now analyze the case of a finite stack of size $L_\parallel$. Now the functional trace in (8) involves a sum over the discrete wavevectors ω_n, given by

$$\omega_n = \frac{2\pi}{L_\parallel} n, \quad n = 0, \pm 1, \pm 2, \cdots. \tag{20}$$

For small λ_0, a series expansion leads to

$$E_1 = \int \mathrm{d}z \, \mathrm{d}^2\sigma \frac{d k_\mathrm{B} T}{2} \varrho_0 e_1, \tag{21}$$

with

$$
\begin{aligned}
e_1 =& \frac{\Lambda^4}{8\pi} \sqrt{\frac{K_0}{B_0}} - \frac{\pi}{12} \sqrt{\frac{B_0}{K_0}} \frac{1}{L_\parallel^2} + \frac{\lambda_0}{8\pi} \sqrt{\frac{K_0}{B_0}} \Lambda^2 + \frac{\lambda_0}{4\pi L_\parallel} \ln\left(\frac{L_\perp^2}{L_\parallel} \sqrt{\frac{B_0}{K_0}} \right) \\
&+ \frac{\lambda_0^2}{64\pi} \sqrt{\frac{K_0}{B_0}} \left[3 - 2\gamma + 2\ln\left(\frac{\lambda_0}{8\pi\Lambda^2} \frac{L_\perp^2}{L_\parallel} \sqrt{\frac{B_0}{K_0}} \right) \right] \\
&+ \sqrt{\pi} \sum_{m=3}^{\infty} \frac{(-1)^{m+1} \lambda_0^m}{m \, 2^{2m} \pi^m} L_\parallel^{m-2} \left(\frac{K_0}{B_0} \right)^{\frac{m-1}{2}} \frac{\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m}{2})} \zeta(m-1).
\end{aligned}
\tag{22}
$$

As in the case of the infinite stack, we absorb the logarithmic divergence by renormalizing the in plane compressibility via Eq. (13), setting τ equal to zero for incompressible membranes. The surface tension receives now an $L_\parallel$-dependent renormalization

$$\sigma = \sigma_0 + \frac{d k_\mathrm{B} T}{16\pi} \sqrt{\frac{K_0}{B_0}} \Lambda^4 - d k_\mathrm{B} T \frac{\pi}{24} \sqrt{\frac{B_0}{K_0}} \frac{1}{L_\parallel^2}, \tag{23}$$

and σ is again set equal to zero to describe a stack of tensionless membranes.

Extremization of the renormalized combined effective action (7) and (21) with respect to ϱ leads again to two possible solutions for the saddle point, namely $\lambda = 0$ or $\lambda = \lambda_{L_\parallel}$, with

$$
\begin{aligned}
& \lambda_{L_\parallel} \left[\ln\left(\frac{\lambda_{L_\parallel}}{\bar{\lambda}} \right) - \frac{1}{2} \right] + \lambda_{L_\parallel} \left[1 - \gamma + \ln\left(\frac{L_\perp^2}{L_\parallel} \frac{1}{8\pi} \sqrt{\frac{B}{K}} \right) \right] \\
& + 32\pi^{3/2} \sum_{m=3}^{\infty} \frac{(-1)^{m+1} \lambda_{L_\parallel}^{m-1}}{m \, 2^{2m} \pi^m} L_\parallel^{m-2} \left(\frac{K}{B} \right)^{\frac{m-2}{2}} \frac{\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m}{2})} \zeta(m-1) = \frac{\sqrt{K}}{a} \left(\frac{1}{T} - \frac{1}{T_{L_\parallel}} \right)
\end{aligned}
\tag{24}
$$

where

$$\frac{1}{T_{L_\parallel}} = \frac{1}{T_{\mathrm{c}}} + \frac{dk_{\mathrm{B}}}{8\pi} \frac{1}{KL_\parallel} \ln\left(\frac{L_\perp^2}{L_\parallel} \sqrt{\frac{B}{K}}\right) \tag{25}$$

is the inverse critical temperature for a stack of size $L_\parallel$.

For $T < T_{L_\parallel}$, Eq. (24) has no solution. In this case, the stack is in the ordered phase, the only available solution for the saddle point being $\lambda = 0$. For $T > T_{L_\parallel}$, there exists a nonzero solution $\lambda_{L_\parallel}$, where the system is in the vertically disordered phase.

Let us now examine the saddle point solutions for ϱ. In the vertically disordered phase where $\lambda = \lambda_{L_\parallel}$ is nonzero, we may expand the effective energy in a small-$L_\parallel$ series. Extremization with respect to $\lambda_{L_\parallel}$ leads to

$$\begin{aligned}
\varrho_+^{-1} = {}& \frac{T}{T_{L_\parallel}} - 1 - \frac{a\lambda_{L_\parallel}T}{\sqrt{K}} \\
& - \frac{dk_{\mathrm{B}}T}{2K}\sqrt{\pi} \sum_{m=3}^{\infty} \frac{(-1)^{m+1}\lambda_{L_\parallel}^{m-1}}{2^{2m}\pi^m}\left(1 - \frac{2}{m}\right) L_\parallel^{m-2}\left(\frac{K}{B}\right)^{\frac{m-1}{2}} \frac{\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m}{2})}\zeta(m-1).
\end{aligned} \tag{26}$$

The positivity of ϱ and the stability of the saddle point again define a maximal temperature, given by

$$\frac{1}{T_{\max}^{L_\parallel}} = \frac{1}{T_{\max}} - \frac{dk_{\mathrm{B}}T}{16\pi KL_\parallel} \ln\left(\frac{16\pi}{dk_{\mathrm{B}}T_{\mathrm{c}}} \frac{\sqrt{BK}}{\bar{\lambda}}\right), \tag{27}$$

above which our assumption that the membranes in the stack are in-plane incompressible cannot be maintained.

In the ordered phase, the situation is more delicate. For $\lambda = 0$, ϱ can be calculated exactly, and we obtain

$$\varrho_-^{-1} = 1 - \frac{dk_{\mathrm{B}}T}{8\pi KL_\parallel} \ln\left[\frac{\sinh\left(\frac{8\pi KL_\parallel}{dk_{\mathrm{B}}T_{\mathrm{c}}}\right)}{\frac{L_\parallel}{2L_\perp^2}\sqrt{\frac{K}{B}}}\right], \tag{28}$$

with an infrared regulator $L_\perp$ equal to the inverse lateral size of the membranes in the stack. If the size $L_\parallel$ of the stack is large, (28) may be approximated by

$$\varrho_-^{-1} \approx 1 - \frac{T}{T_{L_\parallel}}. \tag{29}$$

For smaller stacks, however, the positivity of ϱ is not guaranteed. For a fixed, but small stack size $L_\parallel$, and for fixed lateral size $L_\perp$ of the membranes in the stack, there is a characteristic temperature defined by

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$$T^* = \frac{8\pi K L_\parallel}{d k_B \ln(16\pi \sqrt{BK} L_\perp^2 / d k_B T_c)}, \tag{30}$$

above which ϱ changes sign, and (28) is no longer applicable. Interestingly, for all $L_\perp$ and for all finite sizes $L_\parallel$ of the stack, the critical temperature $T_{L_\parallel}$ is lower than T^*, so that the vertical melting still occurs. The behavior of ϱ is depicted in Fig. 2.
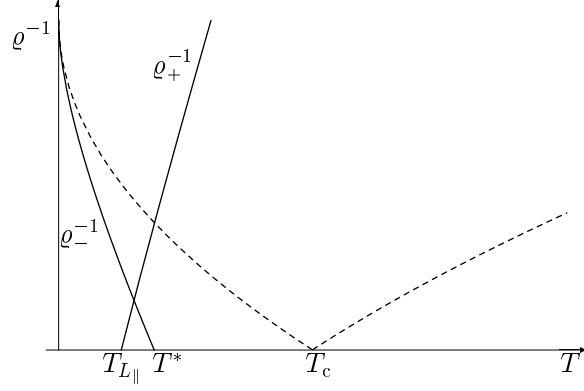

FIG. 2. Behavior of ϱ^{-1} as a function of T. The solid lines indicate the solutions of the saddle point for ϱ^{-1} for a finite stack. Above $T_{L_\parallel}$, ϱ^{-1} is given by (26), and below $T_{L_\parallel}$ by (28). The dashed lines indicate the behavior of ϱ^{-1} for an infinite stack.

Note that Eq. (30) reflects the existence of a characteristic horizontal length scale. At fixed temperature $T_{L_\parallel} < T < T^*$, and for membranes of lateral size $L_\perp$ smaller than

$$L_p = \Lambda^{-1} \exp\left(\frac{4\pi K L_\parallel}{d k_B T}\right), \tag{31}$$

the height fluctuations of the individual membranes are not strong enough to destroy the ordered phase. The characteristic length L_p corresponds to the the de Gennes-Taupin persistence length ξ_p [7] of the individual membranes, below which crumpled membranes appear flat.

D. Finite number of membranes

Until now we have performed our calculations in the somewhat unphysical continuum approximation, by letting the interlayer separation l be very small making the number of membranes in the stack very large. Let us now investigate the properties of the stack for a fixed number $N + 1$ of membranes at a finite interlayer distance l.

For this purpose, we replace the continuum derivative ∂_z^2 in the z-direction with the discrete gradient operator ∇^2, whose eigenvalues are given by

$$\nabla^2 g = \frac{2(1 - \cos \omega_n l)}{l^2} g, \tag{32}$$

where g is some test function. The discrete wavevectors ω_n are now given by

$$\omega_n = \frac{n\pi}{Nl}, \ n = 1, 2, \cdots, N. \tag{33}$$

For small interlayer separation l, the free energy is given by (21) with

$$\begin{aligned}
e_1 &= \frac{\Lambda^4}{8\pi}\sqrt{\frac{K_0}{B_0}} + \frac{1}{Nl^2}\sqrt{\frac{B_0}{K_0}} + \frac{\lambda_0}{8\pi}\sqrt{\frac{K_0}{B_0}}\Lambda^2 + \frac{\lambda_0}{4\pi Nl}\left[-\ln N + 2N \ln\left(l\sqrt{\frac{K_0}{B_0}}\Lambda^2\right)\right] \\
&+ \frac{\lambda_0^2}{64\pi}\sqrt{\frac{K_0}{B_0}}\left[1 - 2\ln\left(\frac{4\Lambda^2}{\lambda_0}\right)\right] + \frac{1}{N\sqrt{\pi}}\sum_{m=2}^{\infty}\frac{(-1)^{m+1}\lambda_0^m}{m\,2^{\frac{3m+1}{2}}}l^{m-2}\left(\frac{K_0}{B_0}\right)^{\frac{m-1}{2}}\frac{\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m}{2})}\bar{\zeta}_N(m-1),
\end{aligned} \tag{34}$$

where we have defined the modified Zeta-function

$$\bar{\zeta}_N(m) = \sum_{n=1}^{N}\frac{1}{\left[1 - \cos\left(\frac{n\pi}{N}\right)\right]^{\frac{m}{2}}}. \tag{35}$$

We proceed by renormalizing the in-plane compressibility via Eq. (13), setting τ equal to zero for incompressible membranes as before. The surface tension receives an l-dependent renormalization

$$\sigma = \sigma_0 + \frac{dk_{\mathrm{B}}T}{16\pi}\sqrt{\frac{K_0}{B_0}}\Lambda^4 - \frac{dk_{\mathrm{B}}T}{2Nl^2}\sqrt{\frac{B_0}{K_0}} \tag{36}$$

and σ is set equal to zero to describe a stack of tensionless membranes, as before. But now the bulk bending rigidity is also modified to

$$K = K_0 - \frac{dk_{\mathrm{B}}T}{4\pi l}\ln\frac{\Lambda^2}{\mu^2}. \tag{37}$$

Note that this renormalization agrees with the known result for a single membrane [8].

The saddle point for λ is now given by $\lambda = 0$ or $\lambda = \lambda_l$, with

$$\begin{aligned}
&\lambda_l\left[\ln\left(\frac{\lambda_l}{\bar{\lambda}}\right) - \frac{1}{2}\right] + 32\sqrt{\pi}\sum_{m=2}^{\infty}\frac{(-1)^{m+1}\lambda_l^{m-1}}{m\,2^{\frac{3m+1}{2}}}l^{m-2}\left(\frac{K}{B}\right)^{\frac{m-2}{2}}\frac{\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m}{2})}\bar{\zeta}_N(m-1) \\
&= \frac{\sqrt{K}}{a}\left(\frac{1}{T} - \frac{1}{T_l}\right)
\end{aligned} \tag{38}$$

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where

$$\frac{1}{T_l} = \frac{1}{T_c} + \frac{dk_B}{8\pi N\kappa}\left\{-\ln N + 2N\left[\ln\left(l\sqrt{\frac{K}{B}}\bar\lambda\right) + \frac{1}{2}\right]\right\} \tag{39}$$

is the inverse l-dependent critical temperature, and κ is the bending rigidity of a single membrane in the stack. The two solutions for λ again imply the existence of two different phases, with a phase transition at the critical temperature T_l, which, as in the perturbative case, depends only weakly on the number of membranes in the stack. The corresponding solutions for ϱ, obtained by extremizing the effective energy with respect to λ, are given by

$$\varrho_-^{-1} = 1 - \frac{T}{T_l}, \tag{40}$$

for $T < T_l$, that is, in the ordered phase, and

$$\varrho_+^{-1} = \frac{T}{T_l} - 1 - \frac{a\lambda_l T}{\sqrt{K}}$$
$$- \frac{dk_B T}{2\sqrt{\pi}NK}\sum_{m=2}^{\infty}\frac{(-1)^{m+1}\lambda_l^{m-1}}{2^{\frac{3m+1}{2}}}\left(1 - \frac{2}{m}\right)l^{m-2}\left(\frac{K}{B}\right)^{\frac{m-1}{2}}\frac{\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m}{2})}\bar\zeta_N(m-1), \tag{41}$$

in the vertically molten phase, where $T > T_l$.

The stability of the saddle point requires a minimum interlayer separation

$$l_{\min} = \mu^{-2}\sqrt{\frac{B}{K}}N^{\frac{1}{2N}}\exp\left(\frac{4\pi\kappa}{dk_B T_c}\right) \tag{42}$$

below which the stack becomes unstable. $l_{\min}$ is inversely proportional to de Genne's penetration depth $\sqrt{K/B}$ [9], which is of the order of the interlayer separation in smectic liquid crystals.

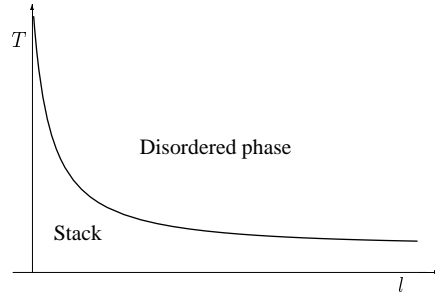The phase diagram of the stack is depicted in Fig. 3

FIG. 3. Qualitative phase diagram in the $l \times T$ plane. The critical line is plotted for $l > l_{\min}$, for a fixed number $N + 1$ of membranes in the stack. As l increases, the critical temperature T_l goes asymptotically to zero.

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E. Properties of the phases

Let us now characterize both phases in more detail. For temperatures higher than T_l, the solution of the saddle point is $\lambda = \lambda_l$. This corresponds to the disordered phase, where the stack melts. The normals to the membranes are uncorrelated beyond a length scale $\lambda_l^{-1/2}$, as can be derived from the expression for the orientational correlation function, which in the limit $N \rightarrow \infty, l \rightarrow 0$, with constant $Nl = L_\parallel$, reads:

$$\langle \partial_i u(\mathbf{x}_\perp, z) \partial_j u(\mathbf{x}'_\perp, z) \rangle \sim \delta_{ij} e^{-\sqrt{\lambda_l}|\mathbf{x}_\perp - \mathbf{x}'_\perp|}. \tag{43}$$

The length scale $\lambda_l^{-1/2}$ may thus be identified with the persistence length ξ_{p} [10].

In the low-temperature phase, the solution of the saddle point is $\lambda = 0$. This corresponds to the ordered lamellar phase as can be seen by examining the orientational correlation function in the planes of the membranes. We find in the limit $N \rightarrow \infty, l \rightarrow 0$, with constant $Nl = L_\parallel$:

$$\langle \partial_i u(\mathbf{x}_\perp, z) \partial_j u(\mathbf{x}'_\perp, z) \rangle \sim \frac{\delta_{ij}}{|\mathbf{x}_\perp - \mathbf{x}'_\perp|^3}. \tag{44}$$

This slow, algebraic fall-off of the correlation function implies that, at large distances, the normal vectors to the membranes are still parallel, so that the surfaces remain flat on the average. The effect of thermal fluctuations is suppressed, and they do not disorder the stack.

Let us now calculate the entropy loss in the ordered phase. By a simple scaling argument [1], this quantity should be inversely proportional to the quadratic interlayer spacing,

$$-T\Delta S = \frac{a}{l^2}, \tag{45}$$

where a is a temperature dependent proportionality constant. As we shall see, as the interlayer distance increases, logarithmic corrections must be added to (45).

The entropy loss in the ordered phase can be computed by calculating the difference between the free energy density of a single, isolated membrane, an the free energy density of the stack. For small values of λ, it is given by

$$-T\Delta S = \frac{1}{2}\sqrt{\frac{B}{K}}\left(1 + \cot\frac{\pi}{4N}\right)\frac{1}{Nl^2} - \frac{\lambda}{2\pi l}\left(1 - \ln\frac{\lambda l}{2}\sqrt{\frac{K}{B}}\right) + \frac{\lambda}{2\pi Nl}\sum_{n=1}^{N}\ln\sin\frac{n\pi}{2N}. \tag{46}$$

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Strictly speaking, the ordered phase corresponds to $\lambda = 0$. In that case, (46) agrees with (45), and we see no correction to the entropy loss. However, if the size of the membranes in the stack is smaller than the characteristic length L_{p}, an ordered phase still exists for small values of λ [see discussion after Eq. (31)], in which case the corrections to the first term in (46) will appear.

IV. CONCLUSIONS

In the limit of large embedding dimension d, we have shown that a stack of tensionless and incompressible membranes melts vertically upon approaching a critical temperature, where the lamellar phase goes over into a disordered phase. In contrast to the low-temperature ordered phase, where the decay of orientational correlations is power-like, the high-temperature disordered phase is characterized by an exponential decay of orientational correlations, with different length scales in the transversal and longitudinal directions.

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